Binomial Distributions

Lesson objectives

- I can recognise conditions that give rise to a binomial probability distribution
- I can make connections among the table, histogram, and algebraic representation of a binomial probability distribution
- I can solve problems involving binomial probability distributions

1.1

Lesson objectives | Teachers' notes | Lesson notes

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Warm up

Many events in games and industry rely on success or failure, and these often can be quantified with probabilities. For example, in the game of Monopoly, success in getting out of jail means rolling doubles, and failure means any other roll. When measuring the fit of car doors, success could be being within a given gap tolerance. Think of other examples where success or failure could be quantified by probability.

Definitions

Binomial Probability Distribution

- A distribution with independent trials whose outcomes are either success or failure
- The random variable is the number of successes in a given number of trials

Example 1

Counting Successes

Two dice are rolled five times. What is the probability that doubles occur twice?

P(double) = $1/6$

P(not a double) = $5/6$

Want two doubles AND three non-doubles

$\Rightarrow \; = {}_5C_2\left(\frac{1}{6}\right)^2 \times {}_3C_3\left(\frac{5}{6}\right)^3$

three non-doubles from three trys

≈ 0.1608

two doubles from fives trys (in any order)

The probability of two doubles in 5 attempts is about 0.1608

Your Turn

A card is repeatedly cut from a deck and replaced each time. What is the probability that, in 10 tries, an ace is cut

a) once? b) three times?

P(Ace) = $1/13$

P(not an ace) = $12/13$

a) P(one ace) $= {}_{10}C_1\left(\frac{1}{13}\right)^1 \times {}_9C_9\left(\frac{12}{13}\right)^9$

≈ 0.3743

b) P(three aces) $= {}_{10}C_3\left(\frac{1}{13}\right)^3 \times {}_7C_7\left(\frac{12}{13}\right)^7$

≈ 0.0312

Probability in a Binomial Distribution

The probability of x successes in n identical independent trials is $P(x) = {}_nC_x p^x q^{n-x}$, where p is the probability of success in an individual trial, and $q = 1 - p$ is the probability of failure.

Each term in the expansion of $(p + q)^n$ represents the probability of one possible outcome in the probability distribution.

${}_nC_x$ is the number of arrangements

p^x is the successes

q^{n-x} is the failures

Expectation for a Binomial Distribution

When determining the expectation for a binomial distribution, you can multiply the number of trials by the probability of success in an individual trial instead of using the standard process.

$E(X) = np$

Example 2

Binomial Distribution

A random number generator provides a number between 1 and 100 over a total of five trials with repetition permitted. Calculate a probability distribution for the number of times a prime number is output.

a) Identify the discrete random variable.
b) Calculate the probability distribution.
c) Verify that the sum of the probabilities is 1.
d) Graph the probability distribution.
e) Describe the shape of the probability histogram.
f) What does $P(5)$ tell you?
g) Calculate the expectation. Interpret its meaning.

a) X = # of primes

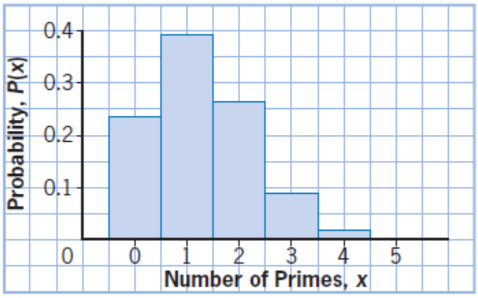
b) Primes from 1 → 100 are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

Total of 25 ⇒ $P(\text{prime}) = \frac{1}{4}$

Number of Primes, x	Probability, $P(x)$	$x \cdot P(x)$
0	${}_5C_0(0.25)^0(0.75)^5 = 0.2373$	0
1	${}_5C_1(0.25)^1(0.75)^4 = 0.3955$	0.3955
2	${}_5C_2(0.25)^2(0.75)^3 = 0.2638$	0.5273
3	${}_5C_3(0.25)^3(0.75)^2 = 0.0879$	0.2637
4	${}_5C_4(0.25)^4(0.75)^1 = 0.0146$	0.0586
5	${}_5C_5(0.25)^5(0.75)^0 = 0.0010$	0.005

c) $0.2373 + 0.3955 + 0.2637 + 0.0879 + 0.0146 + 0.0010 = 1$

d)



e) Graph is somewhat bell-shaped, skewed to the right (more spread out) with a mode of one.

f) $P(5) = 0.001$ which means getting all 5 selections being prime is very unlikely (0.1%)

g) $E(x) = np$
$= 5(0.25)$
$= 1.25$

Expect 1.25 primes on average from 5 numbers

Your Turn

A family has six children. Consider a probability distribution for the number of girls in the family.

a) Identify the discrete random variable.

b) Calculate the probability distribution.

c) Verify that the sum of the probabilities is 1.

d) Graph the probability distribution. Compare the shape of the probability histogram to the one in Example 2.

e) Calculate the expectation. Interpret its meaning.

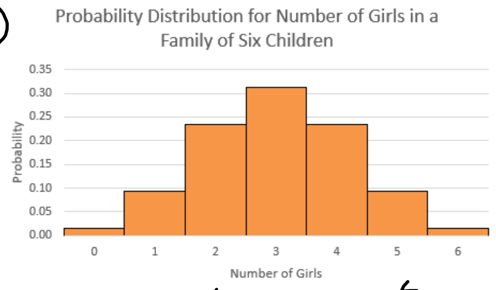
a) X = # of girls

b)

Number of Girls, x	Probability, $P(x)$	$x \cdot P(x)$
0	${}_6C_0(0.5)^0(0.5)^6 = 0.015\,625$	0
1	${}_6C_1(0.5)^1(0.5)^5 = 0.093\,75$	0.093 75
2	${}_6C_2(0.5)^2(0.5)^4 = 0.234\,375$	0.468 75
3	${}_6C_3(0.5)^3(0.5)^3 = 0.3125$	0.9375
4	${}_6C_4(0.5)^4(0.5)^2 = 0.234\,375$	0.9375
5	${}_6C_5(0.5)^5(0.5)^1 = 0.093\,75$	0.468 75
6	${}_6C_6(0.5)^6(0.5)^0 = 0.015\,625$	0.093 75

c) $0.015625 + 0.09375 + 0.234375 + 0.3125 + 0.234375 + 0.09375 + 0.015625 = 1$

d)



Perfect shape, not skewed, mode of 3

e) $E(x) = np$
$= 6(0.5)$
$= 3$

On average, you would expect 3 girls from a family with 6 children

Example 3

Apply the Binomial Distribution

The failure rate is 5% in the initial production run of a new computer chip. A quality control inspector selects 30 chips for testing.

a) What is the probability that more than two of them are defective?

b) What is the expected number of defective chips?

$p = 0.05$
$q = 0.95$
$n = 30$

a) $P(>2) = 1 - P(0) - P(1) - P(2)$

$= 1 - {}_{30}C_0(0.05)^0(0.95)^{30} - {}_{30}C_1(0.05)^1(0.95)^{29} - {}_{30}C_2(0.05)^2(0.95)^{28}$

$= 1 - 0.2146 - 0.3389 - 0.2586$

$\approx 0.1878 \Rightarrow$ 18.78% chance of more than 2 chips being defective

b) $E(x) = np$
$= 30(0.05)$
$= 1.5$

$\Rightarrow$ On average, we would expect there to be 1.5 defective chips from the 30 selected.

Your Turn

With a certain set of atmospheric conditions, the probability of rain is 40%. During a one-month period, eight days had those conditions.

a) What is the probability that it rained on fewer than six of those days?

b) What is the expected number of rainy days?

$p = 0.4$
$q = 0.6$
$n = 8$

a) $P(\text{fewer than 6 days}) = 1 - P(6) - P(7) - P(8)$

$= 1 - {}_8C_6(0.4)^6(0.6)^2 - {}_8C_7(0.4)^7(0.6)^1 - {}_8C_8(0.4)^8(0.6)^0$

$= 1 - 0.0413 - 0.0079 - 0.0007$

$= 0.9501$ $\Rightarrow$ 95.01% chance that it rained on fewer than 6 of the 8 days.

b) $E(x) = np$
$= 8(0.4)$
$= 3.2$

$\Rightarrow$ On average, we would expect 3.2 rainy days from the 8 with those atmospheric conditions.

Key Concepts

- A binomial distribution has a specific number of identical independent trials in which the result is success or failure.
- You can represent a binomial distribution using a table, a histogram, and a formula.
- The probability of x successes in n independent trials is $P(x) = {}_nC_x p^x q^{n-x}$, where p is the probability of success in an individual trial, and $q = 1 - p$ is the probability of failure.
- The expectation for the binomial distribution is $E(X) = np$.

R1. The formula for the binomial distribution is $P(x) = {}_nC_x p^x q^{n-x}$. What is the purpose of the ${}_nC_x$ coefficient?

In the binomial distribution, the coefficient ${}_nC_x$ represents the total number of ways that each of the number of successes can happen.

R2. In a binomial distribution, how are p and q related? Include an example using a deck of cards.

In a binomial distribution, p and q represent the probability of success and failure, respectively. They are related by $p = 1 - q$. For example, the probability of drawing a club (success) from a deck of cards is 0.25, while the probability of not drawing a club (failure) is 0.75.

R3. About 11% of Canadians are left-handed. A newspaper columnist interpreted this to mean that there is an 11% chance that any one of the newspaper's 25 reporters will be left-handed. Discuss the accuracy of this statement.

The statement that about 11% of Canadians are left-handed means that in a group of 25 people, the expected number of left-handed people is 2.75. However, the probability of one person in 25 being left-handed is ${}_{25}C_1(0.11)^1(0.89)^{24}$, or about 16.78%.