Uniform Distributions

Lesson objectives

- I can solve problems involving uniform probability distributions

1.1 Lesson objectives | Teachers' notes | Lesson notes

MHR Page 157 #s 1 - 8

Warm up

On a TV game show, Allie has the option of taking home $750 or guessing which one of 26 briefcases contains $1 000 000.

- What are her chances of winning?
- Is each briefcase equally likely to hold the money? What would you do?
- How would all of this change if she were allowed a second chance after checking the contents of five briefcases?

Definitions

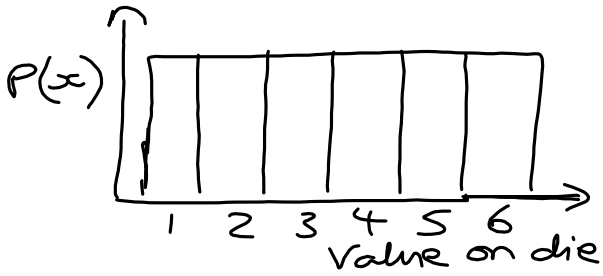
Uniform Distribution

- Occurs when, in a single trial, all outcomes are equally likely
- For all outcomes, x, $P(x) = \frac{1}{n}$, where n is the number of possible outcomes in the experiment

Investigate on Page 154

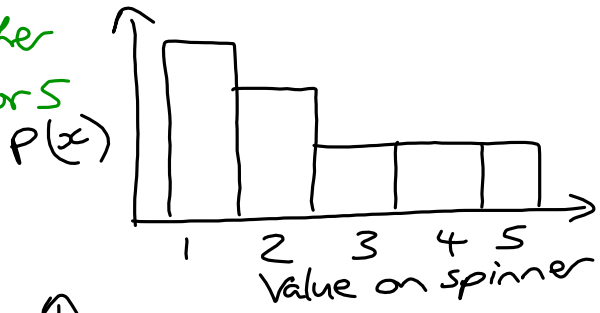
a) the upper face on a single roll of a die

We can get either a 1, 2, 3, 4, 5, or 6



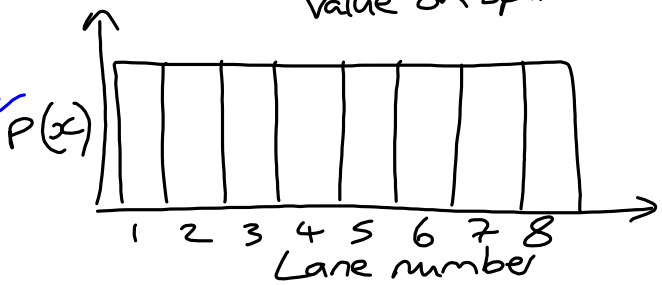
b) the result of a single spin of the spinner

We can get either a 1, 1, 1, 2, 2, 3, 4, or 5



c) the position a person could be assigned when the eight runners in a race are randomly assigned a starting lane from lanes 1 to 8

runner could get either lane 1, 2, 3, 4, 5, 6, 7, or 8



Example 1

Uniform Distribution

A calculator has been programmed to generate a random number between 1 and 5.

a) Classify this distribution.
b) Calculate the probability distribution.
c) Sketch a graph of the distribution. Comment on the shape of the graph.
d) Calculate the expectation. Interpret its meaning.

a) This is a uniform distribution because there is a single trial and each outcome is equally likely.

b)

Random #, x	$P(x)$	$x \cdot P(x)$
1	1/5	1/5
2	1/5	2/5
3	1/5	3/5
4	1/5	4/5
5	1/5	5/5

c)

All the bars are equal

d) $E(x) = \sum_{x=1}^{5} x \cdot P(x)$

$= \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + \frac{5}{5}$

$= \frac{15}{5} = 3$

The predicted average value of the random number is 3.

Your Turn

A screen saver has been programmed to draw a circle with a randomly chosen radius of integer length between 1 and 8 cm.

a) Is the probability distribution of areas uniform? Explain.

b) Calculate the probability distribution.

c) Sketch a graph of the distribution. Comment on the shape of the graph.

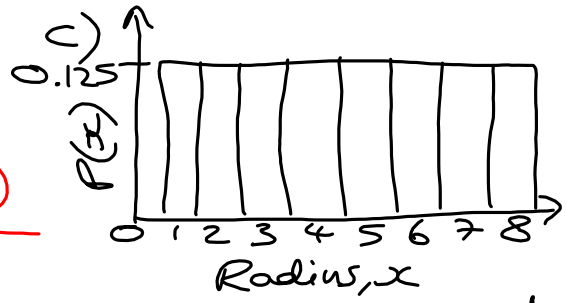
d) Calculate the expectation. Interpret its meaning.

a) Yes. Each radius has an equally likely chance of being generated.

b)

Radius, x	$P(x)$	$x \cdot P(x)$
1	1/8	1/8
2	1/8	2/8
3	1/8	3/8
4	1/8	4/8
5	1/8	5/8
6	1/8	6/8
7	1/8	7/8
8	1/8	8/8

c)



All the bars are equal

d) $E(x) = \sum_{x=1}^{8} x \cdot P(x)$

$= \frac{1}{8} + \frac{2}{8} + \frac{3}{8} + \ldots + \frac{7}{8} + \frac{8}{8}$

$= \frac{36}{8} = 4.5$

The expected radius is 4.5cm

Example 2

Fair Game

A game involves rolling a die. A player who rolls an even number receives points equal to two times the face value of the die. If the player rolls an odd number, the player loses three times the face value of the die. Is this a fair game?

Roll	Points, x	$P(x)$	$x \cdot P(x)$
1	−3	1/6	−3/6
2	4	1/6	4/6
3	−9	1/6	−9/6
4	8	1/6	8/6
5	−15	1/6	−15/6
6	12	1/6	12/6

$E(x) = \sum_{x=1}^{6} x \cdot P(x)$

$= -\frac{3}{6} + \frac{4}{6} - \frac{9}{6} + \frac{8}{6} - \frac{15}{6} + \frac{12}{6}$

$= -\frac{3}{6} = -0.5$ points

A fair game would have $E(x) = 0$

$\Rightarrow$ Not a fair game because would expect to lose on average, 0.5 points per turn.

Your Turn

A spinner has eight equally spaced sectors labelled from 1 to 8. In a particular game, a player wins points equal to double the sector's face value if a power of two is spun. For all other spins, the player loses the face value of the spin. Is this a fair game?

Value	Points, x	$P(x)$	$x \cdot P(x)$
1	−1	1/8	−1/8
2	4	1/8	4/8
3	−3	1/8	−3/8
4	8	1/8	8/8
5	−5	1/8	−5/8
6	−6	1/8	−6/8
7	−7	1/8	−7/8
8	16	1/8	16/8

$$E(x) = \sum_{x=1}^{8} x \cdot P(x)$$

$$= -\frac{1}{8} + \frac{4}{8} - \frac{3}{8} + \frac{8}{8} - \frac{5}{8} - \frac{6}{8} - \frac{7}{8} + \frac{16}{8}$$

$$= \frac{6}{8}$$

$$= 0.75$$

Again a fair game would have $E(x) = 0$

$\Rightarrow$ This is not a fair game because players would expect to win 0.75 points each turn, on average

Key Concepts

- A uniform distribution occurs when, in a single trial, all outcomes are equally likely.
- For a uniform distribution, $P(x) = \frac{1}{n}$, where n is the number of possible outcomes in the experiment.
- When calculating expectation for a uniform distribution, you can factor $\frac{1}{n}$ to make the calculations easier: $E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i$
- When calculating expectation, you can calculate the sum of the numbers from 1 to n using the expression $\frac{n(n+1)}{2}$.
- The expected outcome of a fair game is equal to 0.

R1. A school board randomly selects students by their student number to take part in a survey. Is this process a uniform distribution? Explain.

Yes, randomly selecting students by their student number is uniform. As each student number is different, they are equally likely to be selected in a single trial.

R2. A spinner has 10 equally spaced sectors. Draw an example of a spinner with

a) a uniform distribution

b) a non-uniform distribution

a) A spinner with a uniform distribution has equally sized sectors

b) A spinner with a non-uniform distribution will have unequally sized sectors

R3. A school raffle has payouts totalling $1500. The school expects 2000 tickets to be sold. Will a price of $2 per ticket give an advantage to the school, to the customers, or will it be a fair game? Justify mathematically.

$$\text{Expected payout} = \frac{\text{value of prizes}}{\text{\# of tickets sold}} = 1500 \div 2000$$

$$= 0.75$$

The expected payout of each ticket is $0.75

The profit per ticket = cost of ticket - payout

= 2.00 - 0.75

= $1.25

This gives an advantage to the school