Review

1. Review Sheet Topics:
   - Exponent Laws
   - Rational Exponents
   - Graphs of Exponential Functions
   - Properties of Exponential Functions
   - Transformations
   - Solving Exponential Functions
   - Growth and Decay Problems

2. Review Questions
   Nelson Page 267 #s 2 - 5, 7 & 9 - 17

	Equation	Graph	Table of Values
Linear	All exponents have a degree of 1	Straight line	1ST diffs are constant
Quadratic	Independent variable has degree of 2	Parabola	2ND diffs are constant
Exponential	Independent variable is the exponent	Slow to rapid increase or decrease	y-values change by a common ratio/factor

Exponent Law #1 - Multiplying Powers

When we multiply terms with the same base we **add the exponents** and keep the base the same.

Example:

a) $(2^2)(2^3)$

$= 2^{2+3}$

$= 2^5$

b) x^4x^6

$= x^{4+6}$

$= x^{10}$

c) $(2x^3)(-x^2)$

$= (2)(x^3)(-1)(x^2)$

$= -2x^{3+2}$

$= -2x^5$

Exponent Law #2 - Dividing Powers

When we divide terms with the same base **we subtract the exponents** and keep the base the same.

Example:

a) $\frac{4^6}{4^4}$

$= 4^{6-4}$

$= 4^2$

b) $\frac{(x)^{12}}{(x)^8}$

$= x^{12-8}$

$= x^4$

c) $\frac{24x^5}{6x^2}$

$24 \div 6 = 4$

$x^5 \div x^2 = x^3$

$\Rightarrow 4x^3$

Exponent Law #3 - Power of a Power Law

When the base of a power is itself a power, **each factor in the base has to have its power multiplied by the power outside the bracket**.

Example:

a) $(x^2)^3$

$= x^{2(3)}$

$= x^6$

b) $(x^3y^4)^2$

$= x^{3(2)}y^{4(2)}$

$= x^6y^8$

c) $(2x^2)^3$

$= 2^3x^{2(3)}$

$= 8x^6$

Exponent Law #4 - Zero Power Law

Anything to the power of 0 is equal to 1.

Example:

a) 3^0

$= 1$

b) x^0

$= 1$

c) $(12x^3y^7)^0$

$= 1$

$2((6x^3y^2)^4)^0 = 2(1)$

$= 2$

Exponent Law #5 - Negative Power Law

A base raised to a negative power is **equivalent to the reciprocal of the same base raised to the positive exponent**.

Example:

a) 3^{-2} b) $\left(\frac{3}{4}\right)^{-3}$ c) $\frac{1}{4^{-2}}$

$= \frac{1}{3^2}$ $= \left(\frac{4}{3}\right)^3$ $= 4^{-(-2)}$

$= \left(\frac{1}{3}\right)^2$ $= 4^2$

Rational Exponents

Using the exponent laws simplify the following:

$4^{\frac{1}{2}} \times 4^{\frac{1}{2}}$ and $27^{\frac{1}{3}} \times 27^{\frac{1}{3}} \times 27^{\frac{1}{3}}$

$= 4^{1/2+1/2} = 4^1 = 4$ $= 27^{1/3+1/3+1/3} = 27^1 = 27$

What value would you assign to $4^{\frac{1}{2}}$ $27^{\frac{1}{3}}$?

$4^{1/2} = 2$ $27^{1/3} = 3$

What about $16^{\frac{1}{2}}$ $8^{\frac{1}{3}}$ $64^{\frac{1}{4}}$?

$16^{1/2} = 4$ $8^{1/3} = 2$ $64^{1/4} = 2.828$

Rational Exponents

Essentially, this has given us a new way to write radicals.

The radical is incorporated into the exponent as the **denominator of the fraction**.

Square root $\longrightarrow$ Exponent = 1/2

Cubed root $\longrightarrow$ Exponent = 1/3

Fourth root $\longrightarrow$ Exponent = 1/4

Rational Exponents

The numerator of the exponent still represents the power.

For example: $8^{\frac{2}{3}}$

$\left(8^{1/3}\right)^2$ or $\left(\sqrt[3]{8}\right)^2$

Things we should know....

$8^{\frac{1}{3}} = 2$ $16^{\frac{1}{4}} = 2$

$27^{\frac{1}{3}} = 3$ $81^{\frac{1}{4}} = 3$

$64^{\frac{1}{3}} = 4$

$125^{\frac{1}{3}} = 5$ $625^{\frac{1}{4}} = 5$

All the perfect squares!!

1^2
2^2
3^2
$\vdots$
98^2
99^2
100^2

Example

Evaluate

a) $64^{\frac{2}{3}}$

$= \left(\sqrt[3]{64}\right)^2$

$= (4)^2$

$= 16$

b) $16^{\frac{2}{3}}$

$= \left(\sqrt[3]{16}\right)^2$

$= (2.5198)^2$

$= 6.3496$

c) $(-27)^{\frac{4}{3}}$

$= \left(\sqrt[3]{-27}\right)^4$

$= (-3)^4$

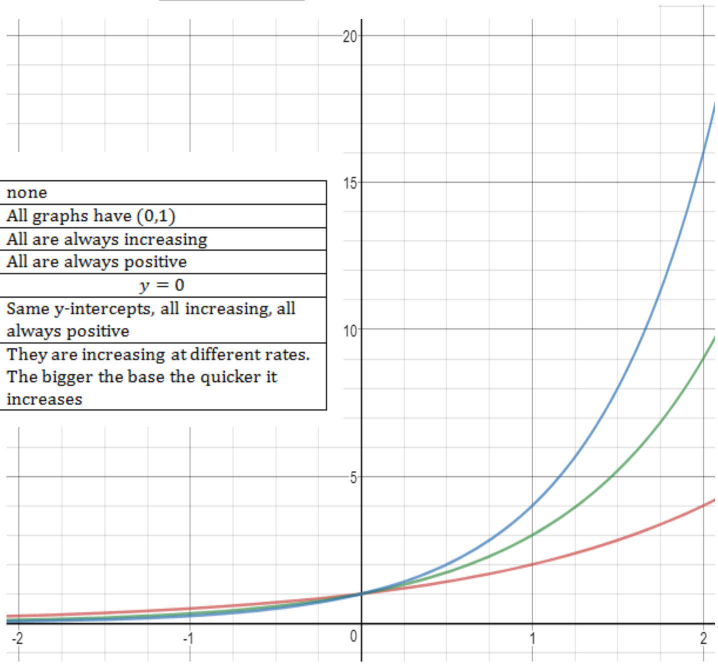
$= 81$

Use the table of values to sketch each of the following graphs.

$y = 2^x$	
x	y
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

$y = 3^x$	
x	y
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9

$y = 4^x$	
x	y
-2	$\frac{1}{16}$
-1	$\frac{1}{4}$
0	1
1	4
2	16

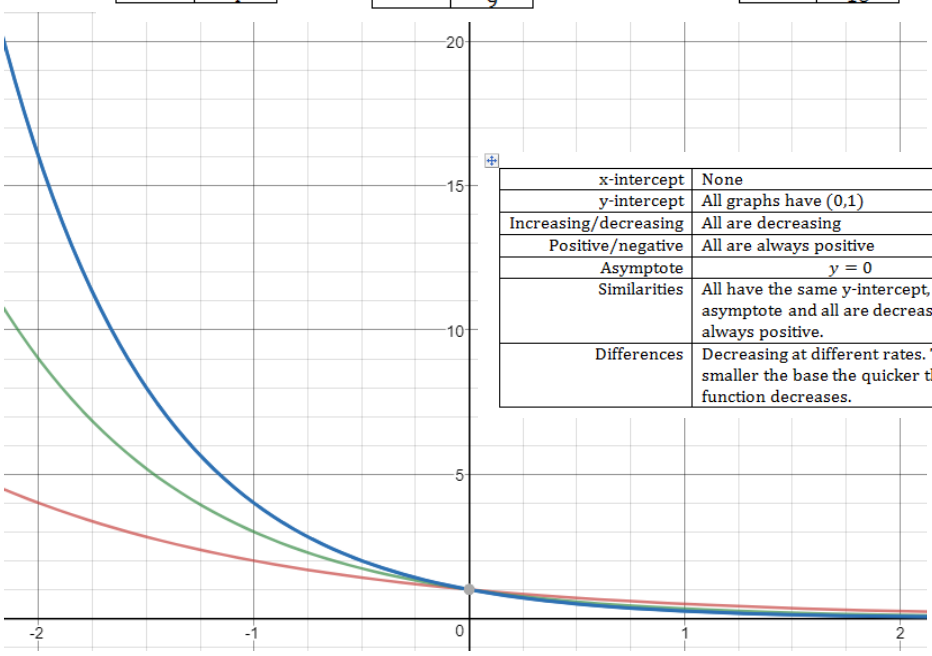


x-intercept	none
y-intercept	All graphs have (0,1)
Increasing/decreasing	All are always increasing
Positive/negative	All are always positive
Asymptote	$y = 0$
Similarities	Same y-intercepts, all increasing, all always positive
Differences	They are increasing at different rates. The bigger the base the quicker it increases

$y = \left(\frac{1}{2}\right)^x$	
x	y
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

$y = \left(\frac{1}{3}\right)^x$	
x	y
-2	9
-1	3
0	1
1	$\frac{1}{3}$
2	$\frac{1}{9}$

$y = \left(\frac{1}{4}\right)^x$	
x	y
-2	16
-1	4
0	1
1	$\frac{1}{4}$
2	$\frac{1}{16}$



x-intercept	None
y-intercept	All graphs have (0,1)
Increasing/decreasing	All are decreasing
Positive/negative	All are always positive
Asymptote	$y = 0$
Similarities	All have the same y-intercept, asymptote and all are decreasing and always positive.
Differences	Decreasing at different rates. The smaller the base the quicker the function decreases.

Domain and Range - The Base Functions

Are there any restrictions on the values of x we can put in the equation?

No $\Rightarrow$ $D = \{x \in \mathbb{R}\}$

Are there any restrictions on the values of y we can calculate from the equation?

Yes $\Rightarrow$ $R = \{y \in \mathbb{R} \mid y > 0\}$

Never reaches zero, so it is not included.

Example

Predict the changes to the graph. Test your theories on desmos!

$$f(x) = 3^x$$

$$f(x) = 3^{x-1} + 4$$

$$g(x) = \left(\frac{1}{4}\right)^x$$

$$g(x) = -2\left(\frac{1}{4}\right)^x - 1$$

HT right 1
VT up 4

Reflect in x-axis
VS factor of 2
VT down 1

Transformations

horizontal stretch/
compression/reflection
(by a factor of 1/k)

vertical stretch/
compression/reflection

$$y = ab^{k(x-d)} + c$$

vertical shift by
c units

base

horizontal shift d units
(x-d) - right
(x+d) - left

Graphing Transformations

We have **TWO** anchor points to graph exponential functions, as well as the asymptote.

1. (0,1) 2. (1,b)

We can apply the transformations to these two points and the asymptote to sketch the graph.

1. (0,1) gives $\left(\left(\frac{1}{k}(0)+d\right), a(1)+c\right)$

2. (1,b) gives $\left(\left(\frac{1}{k}(1)+d\right), a(b)+c\right)$

aymptote = c

Example

State the transformations to each of the following functions.

$$y = 2(3)^{-x} + 1$$

VS factor of 2
Reflect in y-axis
VT up 1

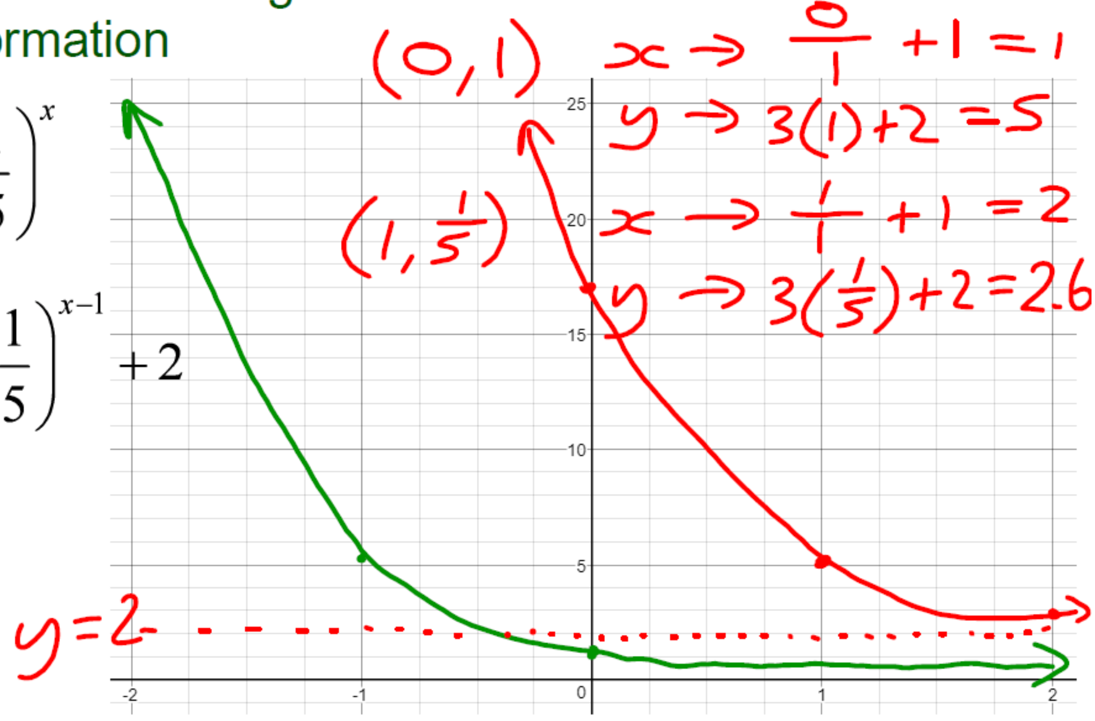
$$y = -\frac{1}{2}^{3(x+2)} + 1$$

Reflect in x-axis
HC factor of $\frac{1}{3}$
HT left 2
VT up 1

Example

Sketch the following base function and the transformation

- $y = \left(\frac{1}{5}\right)^{x}$
- $y = 3\left(\frac{1}{5}\right)^{x-1} + 2$



Example

A 200g sample of radioactive polonium-210 has a half-life of 138 days. This means that every 138 days, the amount of polonium left in a sample is half of the original amount. The mass of polonium, in grams, that remains after t days can be modelled by

$$m(t) = 200(0.5)^{t/138}$$

a) Determine the mass that remains after 5 years.

$t = 5(365)$

$t = 1825$

$\Rightarrow m(1825) = 200(0.5)^{\frac{1825}{138}}$

$= 0.02g$

b) How long does it take for the sample of 200g to decay to 12.5 grams?

$200g \rightarrow 100g$

$100g \rightarrow 50g$

$50g \rightarrow 25g$

$25g \rightarrow 12.5g$

4 half-life cycles

$\Rightarrow t = 4(138)$

$t = 552$ days

Equivalent Exponential Expressions

We saw before with polynomial expressions that we can prove expressions are **NOT** equal by giving a counter-example, but we need to show the equations are the same to prove that they are equivalent.

We have the same ideas for exponential functions.

Eg Show $2^{2x} = 4^x$

x	2^{2x}	4^x
0	$2^{2(0)} = 1$	$4^0 = 1$
1	$2^{2(1)} = 4$	$4^1 = 4$
2	$2^{2(2)} = 16$	$4^2 = 16$
3	$2^{2(3)} = 64$	$4^3 = 64$

Looking at the table of values on the left we can see that many values are the same, but we need to prove the equations are.

Equivalent Bases

First we need to show that they have the same base.

2^{2x} 4^{x}

We know that $4 = 2^2$ therefore we could have a base of 2

We then substitute so they are expressed as the same base

$2^{2x} = (2^2)^x$

Using the power of a power law we

$2^{2x} = 2^{2x}$ ← power of a power law

and we can see that we have equivalent expressions because the equations are the same

Example

Show that the following are equivalent:

a) $3^{6x} = 9^{3x}$

Since $9 = 3^2$

$\Rightarrow 3^{6x} = (3^2)^{3x}$

$3^{6x} = 3^{6x}$

b) $8^{4x} = 16^{3x}$

$8 = 2^3$, $16 = 2^4$

$\Rightarrow (2^3)^{4x} = (2^4)^{3x}$

$2^{12x} = 2^{12x}$

c) $(\sqrt{3})^2 = (1/3)^{-1}$

$\sqrt{3} = 3^{1/2}$, $\frac{1}{3} = 3^{-1}$

$\Rightarrow (3^{1/2})^2 = (3^{-1})^{-1}$

$3^1 = 3^1$

d) $3^{2x} = 81^{0.5x}$

$81 = 3^4$

$\Rightarrow 3^{2x} = (3^4)^{0.5x}$

$3^{2x} = 3^{2x}$

Solving Exponential Functions

We can also use this method of equivalent expressions to solve problems.

If we know the expressions are equivalent and we have the same base, then the exponents must be the same

Eg Solve for x, given that $3^{2x} = 81$

$3^4 = 81$

$\Rightarrow 3^{2x} = 3^4$

If bases are equal, then exponents are equal

So, $\Rightarrow \frac{2x}{2} = \frac{4}{2}$

$x = 2$

Example

Solve for x

a) $4^x = 16$

$16 = 4^2$

$\Rightarrow 4^x = 4^2$

So, $x = 2$

b) $9^{2x} = 27^3$

$9 = 3^2$, $27 = 3^3$

$\Rightarrow (3^2)^{2x} = (3^3)^3$

$3^{4x} = 3^9$

So, $\frac{4x}{4} = \frac{9}{4}$

$x = 2.25$

c) $4^{2x+1} = 2^4$

$4 = 2^2$

$\Rightarrow (2^2)^{2x+1} = 2^4$

$2^{4x+2} = 2^4$

So, $4x + 2 = 4$

$\frac{4x}{4} = \frac{2}{4}$

$x = 0.5$

Solving using Factoring

We also could come across scenarios that require us to use quadratic factoring.

Example

Find all possible values of x for $2^{x^2+3x} = \frac{1}{4}$

$\frac{1}{4} = 4^{-1} = (2^2)^{-1} = 2^{-2}$

$\Rightarrow 2^{x^2+3x} = 2^{-2}$

So, $x^2 + 3x = -2$

$x^2 + 3x + 2 = 0$

$(x+1)(x+2) = 0$

$\Rightarrow x = -1, x = -2$

Half-Life Problems

Radioactive material decays (half-life problems) according to the relationship

$$A(t) = A_o(0.5)^{t/h}$$

where A(t) is the mass of the decayed material
A_o is the mass of the original material
t is the time
h is the half-life

When solving these problems for time we need to use the strategies involved with equivalent expressions.

Exponential Growth

This can be modelled by

$$A(t) = A_o(b)^t$$

where A(t) is the amount after growth
A_o is the original amount
b is the growth rate per time period
t is the number of growth periods

NOTE: For growth problems, b **MUST** be greater than 1.

i.e. If growing by 5%, then b = 1 + 0.05 = 1.05

Exponential Decay

This can be modelled by

$$A(t) = A_o(b)^t$$

where A(t) is the amount after growth
A_o is the original amount
b is the growth rate per time period
t is the number of growth periods

NOTE: For decay problems, b **MUST** be less than 1.

i.e. If decaying by 5%, then b = 1 - 0.05 = 0.95
If it decays by 5% then 95% remains.

Example

Bacteria quadruple in population every 6 hours

a) If you had 100 bacteria in a petrie dish for 100 hours how many bacteria would be present?

$A_0 = 100$

$b = 4$

$t = 100/6$

[# of growth cycles]

$A\left(\frac{100}{6}\right) = 100(4)^{100/6}$

$A = 1.08 \times 10^{12}$

b) If you started with 100 bacteria and after a certain time had 25600, how long would have the bacteria been present?

$$\frac{25600}{100} = \frac{100(4)^{t/6}}{100}$$

$$256 = 4^{t/6} \qquad [256 = 4^4]$$

$$4^4 = 4^{t/6}$$

So, $4 = t/6 \Rightarrow 6(4) = t = 24$ hours

Example

Bunter bought a car for \$18000. The value of the car depreciates by 20% every year.

a) Write the equation that represents this relationship.

$A_0 = 18000$

$b = \left(1 - \frac{20}{100}\right) = 0.80$

$t = \#$ of years

$A(t) = 18000(0.80)^t$

b) If Bunter owns the car for 4 years, what is the car's value?

Find $A(4)$

$$\Rightarrow A(4) = 18000(0.80)^4$$

$$= \$7372.80$$