Combinations Review

Learning Goals

Section	After this section, I can
3.1	• recognize the advantages of different counting techniques • make connections between situations that involve permutations and combinations
3.2	• recognize the advantages of using permutations and combinations over other counting techniques • apply combinations to solve counting problems • express combinations in standard notation, $C(n, r)$, ${}_nC_r$, $\binom{n}{r}$
3.3	• distinguish situations that use permutations from those that use combinations • solve counting problems using the rule of sum and the fundamental counting principle
3.4	• make connections between combinations and Pascal's triangle
3.5	• solve probability problems using counting principles

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Solutions

1. How many ways are there to arrange four boys and three girls in a line for a photo if the girls must be in order of height from shortest to tallest?

4 boys and 3 girls = Total of 7

Girls must be in order, so treat as identical objects.

$\Rightarrow \frac{7!}{3!} = 840$ ways

2. How many arrangements are there of the letters in each word?

a) ANAGRAM

7 letters, 3 As

$\Rightarrow \frac{7!}{3!} = 840$ arrangements

b) EXPRESSIONS

11 letters, 2 Es, 3 Ss

$\Rightarrow \frac{11!}{2!3!} = 3,326,400$ arrangements

c) ENGINEERING

11 letters, 3 Es, 3 Ns, 2 Gs, 2 Is

$\Rightarrow \frac{11!}{3!3!2!2!} = 277,200$ arrangements

3. The rooms in Kendra's apartment have 14 walls in total. She has enough paint to cover 10 of these walls in one colour and the rest in another colour. In how many ways could Kendra paint her apartment?

14 walls, 10 in one colour, 4 in another

$$\Rightarrow \frac{14!}{10!\,4!} = 1001 \text{ different ways}$$

OR

$$\underbrace{{}_{14}C_{10}}_{\text{10 from 14 in one colour}} \times \underbrace{{}_{4}C_{4}}_{\text{four from remaining four in another}} = 1001 \times 1 = 1001$$

4. a) Determine the value for r that has the greatest number of combinations.

i) $C(8, r)$ ii) $C(10, r)$ iii) $C(7, r)$ iv) $C(15, r)$

Think in terms of Pascal's triangle. You want the middle term for row "n".

i) Row 8 $\rightarrow$ 0 – 8
Middle term = 4
$\Rightarrow r = 4$

ii) Row 10 $\rightarrow$ 0 – 10
Middle term = 5
$\Rightarrow r = 5$

iii) Row 7 $\rightarrow$ 0 – 7
Middle term = 3 or 4
$\Rightarrow r = 3$ or 4

iv) Row 15 $\rightarrow$ 0 – 15
Middle term = 7 or 8
$\Rightarrow r = 7$ or 8

b) Generalize your findings.

When n is even, greatest number of combinations occur when $r = \frac{n}{2}$

When n is odd, greatest number of combinations occur when $r = \frac{n}{2} \pm 0.5$

5. A school choir has 10 sopranos, eight altos, seven tenors, and five basses. How many ways are there to select

a) an octet of three sopranos, two altos, two tenors, and a bass?

b) a barbershop quartet of two tenors and two basses?

a)
Sopranos $= {}_{10}C_3$
Altos $= {}_8C_2$
Tenors $= {}_7C_2$
Bass $= {}_5C_1$
$\Rightarrow {}_{10}C_3 \times {}_8C_2 \times {}_7C_2 \times {}_5C_1$
$= 120 \times 28 \times 21 \times 5$
$= 352{,}800$ ways

b)
Tenors $= {}_7C_2$
Basses $= {}_5C_2$
$\Rightarrow {}_7C_2 \times {}_5C_2$
$= 21 \times 10$
$= 210$ ways

6. A committee of four people is to be chosen from a list of 10 people.

a) In how many ways could this be done?

b) Rewrite the problem so that it requires permutations in its solution.

c) Solve the new problem.

a) 4 from 10
$\Rightarrow {}_{10}C_4$
$= 210$ ways

one possible question

b) Choosing a Chairman, President, Treasurer and Secretary from a committee of ten people.

c) 4 from 10
$\Rightarrow {}_{10}P_4$
$= 5040$

7. In how many ways could five different envelopes be distributed into three mailboxes?

Either 3 in one, 1 in the others ①
or 2 in one, 2 in one, 1 in the other ②

Case ①

$_5C_3 \times {_2C_1} \times {_1C_1} \times 3! = 120$

3 from 5 in one mailbox

1 from remaining 2 in one mailbox

remaining one

# of arrangements of the 3 mailboxes

Case ②

$_5C_2 \times {_3C_2} \times {_1C_1} \times 3! = 180$

2 from 5

2 from remaining 3

remaining one

8. You have one each of $5, $10, $20, $50, and $100 bills in your wallet. How many different sums of money could you form by reaching into your wallet and pulling out some bills?

Have to pull out at least one bill, so we don't need the null set.

$\Rightarrow {_5C_1} + {_5C_2} + {_5C_3} + {_5C_4} + {_5C_5}$

Adding because we could have 1 bill, or 2 bills, or 3 bills, or 4 bills, or all 5 bills.

$= 5 + 10 + 10 + 5 + 1$

$= 31$ different sums

OR indirect method

$2^5 - 1$

pick/don't pick

# of bills

remove the null set

$= 32 - 1$

$= 31$ different sums

9. You are selecting an 8-character password using 26 letters and numbers 0 through 9. In how many ways could your password contain

a) at least two letters?

b) at least two numbers?

c) at least two letters and two numbers?

Total arrangements $= {}_{36}C_8$ ← # of characters

36 = 26 letters + 10 digits

a) At least 2 letters ⇒ Not 0 letters OR 1 letter

0 letters

$\Rightarrow {}_{26}C_0 \times {}_{10}C_8$

$= 1 \times 45$

$= 45$ ways

1 letter

$\Rightarrow {}_{26}C_1 \times {}_{10}C_7$

$= 26 \times 120$

$= 3120$ ways

⇒ At least 2 letters

= All arrangements − 0 letters − 1 letter

$= {}_{36}C_8 - 45 - 3120$

$= 30{,}260{,}340 - 45 - 3120$

$= 30{,}257{,}175$ ways

9. You are selecting an 8-character password using 26 letters and numbers 0 through 9. In how many ways could your password contain

a) at least two letters?

b) at least two numbers?

c) at least two letters and two numbers?

At least 2 numbers ⇒ NOT 0 numbers OR 1 number

0 numbers

$= {}_{26}C_8 \times {}_{10}C_0$

$= 1{,}562{,}275 \times 1$

$= 1{,}562{,}275$

1 number

$= {}_{26}C_7 \times {}_{10}C_1$

$= 657{,}800 \times 10$

$= 6{,}578{,}000$

Total arrangements of at least 2 numbers

$= {}_{36}C_8$ − 0 numbers − 1 number

$= 30{,}260{,}340 - 1{,}562{,}275 - 6{,}578{,}000$

$= 22{,}120{,}065$

9. You are selecting an 8-character password using 26 letters and numbers 0 through 9. In how many ways could your password contain

a) at least two letters?

b) at least two numbers?

c) at least two letters and two numbers?

At least 2 letters and 2 numbers

$\Rightarrow$ NOT 0 letters OR 0 numbers
OR 1 letter OR 1 number

All arrangements − 0 letters − 0 numbers − 1 letter − 1 number

= 30,260,340 − 45 − 1,562,275 − 3120 − 6,578,000

= 22,116,900

10. Some entries of two rows of Pascal's triangle are given. Determine the unknown entries.

330 462 b
a 924

$a = 330 + 462$
$a = 792$

$462 + b = 924$
$b = 924 - 462$
$b = 462$

11. A circle is drawn with n points situated on its circumference.

a) Using these points as vertices, how many quadrilaterals can be formed if

i) $n = 4$?

ii) $n = 5$?

iii) $n = 6$?

Need to use 4 vertices

i) ${}_4C_4 = 1$

ii) ${}_5C_4 = 5$

iii) ${}_6C_4 = 15$

b) Identify these numbers in Pascal's triangle. Describe their location.

c) Relate these numbers to combinations in terms of ${}_nC_r$.

d) How many quadrilaterals can be formed if $n = 12$?

b)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 5 1

They are in diagonal 4 $(r = 4)$

c) Row 4 $\Rightarrow r = 4$

$\Rightarrow {}_nC_4$

d) $n = 12$

$\Rightarrow {}_{12}C_4 = 495$ quadrilaterals

12. Determine the row in Pascal's triangle that has a sum of

a) 512

$2^n = 512$

$\Rightarrow n = 9$

b) 4096

$2^n = 4096$

$\Rightarrow n = 12$

Can solve using logs

$2^n = 512$

$$\frac{n \log 2}{\log} = \frac{\log 512}{\log}$$

$n = 9$

$2^n = 4096$

$$\frac{n \log 2}{\log 2} = \frac{\log 4096}{\log 2}$$

$n = 12$

13. Stephen's school is four blocks west and seven blocks south of his home. Use two methods to determine the number of routes he could take to school, travelling west or south at all times.

Travels $4+7=11$ blocks in total

$n(\text{routes}) = n(\text{west}) \times n(\text{south})$
$= {}_{11}C_4 \times {}_7C_7$
$= 330 \times 1$
$= 330$ routes

Equally...
$n(\text{south}) \times n(\text{west})$
$= {}_{11}C_7 \times {}_4C_4$
$= 330 \times 1$
$= 330$

14. Use Pascal's triangle and combinations to expand and simplify.

a) $(a+b)^5$

$= {}_5C_0 a^5b^0 + {}_5C_1 a^4b^1 + {}_5C_2 a^3b^2 + {}_5C_3 a^2b^3 + {}_5C_4 a^1b^4 + {}_5C_5 a^0b^5$

$= 1a^5b^0 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

b) $(2x+y)^4$

$= {}_4C_0(2x)^4(y)^0 + {}_4C_1(2x)^3(y)^1 + {}_4C_2(2x)^2(y)^2 + {}_4C_3(2x)^1(y)^3 + {}_4C_4(2x)^0(y)^4$

$= 1(16x^4)(1) + 4(8x^3)(y) + 6(4x^2)(y^2) + 4(2x)(y^3) + 1(1)(y^4)$

$= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

15. A poker hand of five cards is dealt from a standard deck.

a) What is the probability that the jack, queen, and king of hearts, but no other hearts, are in the same hand?

b) What is the probability that the hand contains five hearts?

$n(\text{Poker hands}) = {}_{52}C_5$

a) J, Q, K Hearts

$= {}_3C_3 \leftarrow$ choosing 3 from 3

No other hearts

$= {}_{39}C_2 \leftarrow$ choosing 2 non-hearts from 39

$P = \dfrac{{}_3C_3 \times {}_{39}C_2}{{}_{52}C_5}$

$= \dfrac{1 \times 741}{2{,}598{,}960}$

$= 0.000285114$

$= 0.0285\ \%$

b) 5 hearts

$= {}_{13}C_5 \leftarrow$ choosing 5 hearts from 13 hearts

$P = \dfrac{{}_{13}C_5}{{}_{52}C_5}$

$P = \dfrac{1287}{2{,}598{,}960}$

$= 0.000495198$

$= 0.0495\ \%$

16. Five students are randomly selected from seven boys and six girls to join a ski trip. What is the probability that

a) all are girls?

b) there are more boys than girls?

7 boys, 6 girls, total of 13

$n(\text{arrangements}) = {}_{13}C_5$

a) $n(\text{all girls}) = {}_6C_5$

$P(\text{all girls}) = \dfrac{{}_6C_5}{{}_{13}C_5} = \dfrac{6}{1287} = \dfrac{2}{429}$

$= 0.00466\ldots$

b) More boys than girls when

$n(\text{Boys}) = 3$ or 4 or 5

3 Boys

$= {}_7C_3 \times {}_6C_2$

$= 35 \times 15$

$= 525$

4 Boys

$= {}_7C_4 \times {}_6C_1$

$= 35 \times 6$

$= 210$

5 Boys

$= {}_7C_5 \times {}_6C_0$

$= 21 \times 1$

$= 21$

$\Rightarrow$ # ways more boys than girls $= 525 + 210 + 21$

$= 756$

$P(\text{more boys than girls}) = \dfrac{756}{1287} = \dfrac{84}{143}$

$\approx 0.587\ldots$

17. A total of 25 photos have been submitted for a photo competition, three of which were submitted by you. From the submissions, five photos will be chosen as finalists.

a) What is the probability that all of your photos will be chosen as finalists?

b) What is the probability that none of your photos will be chosen as a finalist?

c) What is the probability that at least one of your photos will be chosen as a finalist?

n(arrangements of finalists) $= {}_{25}C_5 = 53130$

a) n(yours all finalists) $= {}_3C_3 \times {}_{22}C_2$
$= 1 \times 231$
$= 231$

$\Rightarrow$ Probability $= \frac{231}{53130} = \frac{1}{230} \approx 0.00435\ldots$

b) n(none of yours finalists) $= {}_3C_0 \times {}_{22}C_5$
$= 1 \times 26334$
$= 26334$

$\Rightarrow$ Probability $= \frac{26334}{53130} \approx 0.4957\ldots$

c) P(at least one chosen) $= 1 - P(\text{none chosen})$
$= 1 - 0.4957\ldots$
$= 0.5043\ldots$