Combinations and Pascal's Triangle

Lesson objectives

- I can make connections between combinations and Pascal's triangle

1.1

Lesson objectives | Teachers' notes | Lesson notes

MHR Page 126 #s 1 - 3, 5, 6 & 13abc

Scholars in many different cultures have known about **Pascal's triangle** for thousands of years. The modern version is attributed to Blaise Pascal, a 17th-century mathematician and philosopher. He discovered numerous patterns in the triangle, including those relating it to combinatorics and probability.

The top rows of Pascal's triangle are shown, along with the term references. The terms are designated by $t_{n,r}$, where n is the row number, starting at zero, and r is the diagonal number, also starting at zero.

- Describe how to generate any given row.
- What are the terms of row 5?

Terms	Row	Term references
1	Row 0	$t_{0,0}$
1 1	Row 1	$t_{1,0}$ $t_{1,1}$
1 2 1	Row 2	$t_{2,0}$ $t_{2,1}$ $t_{2,2}$
1 3 3 1	Row 3	$t_{3,0}$ $t_{3,1}$ $t_{3,2}$ $t_{3,3}$
1 4 6 4 1	Row 4	$t_{4,0}$ $t_{4,1}$ $t_{4,2}$ $t_{4,3}$ $t_{4,4}$
1 5 10 10 5 1		

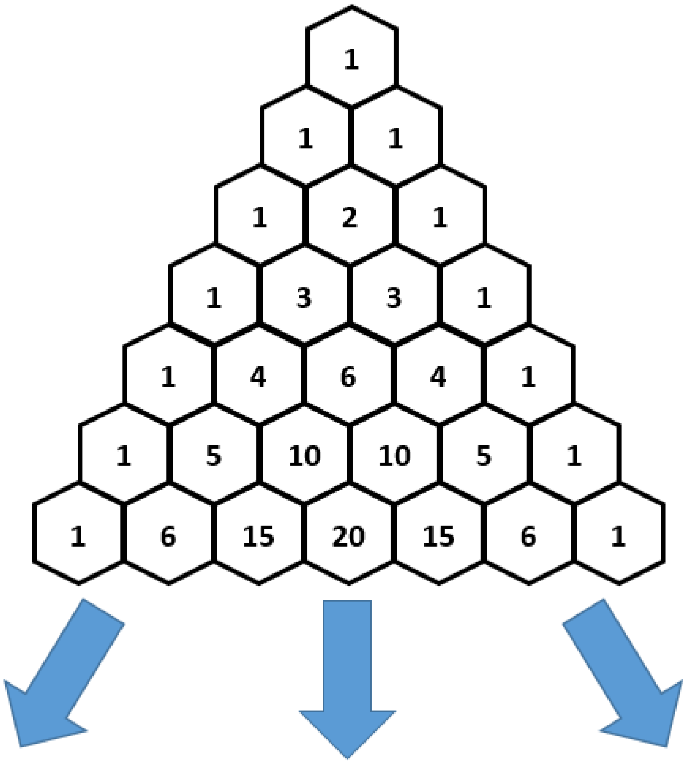
1+4 → 5, 6+4 → 10

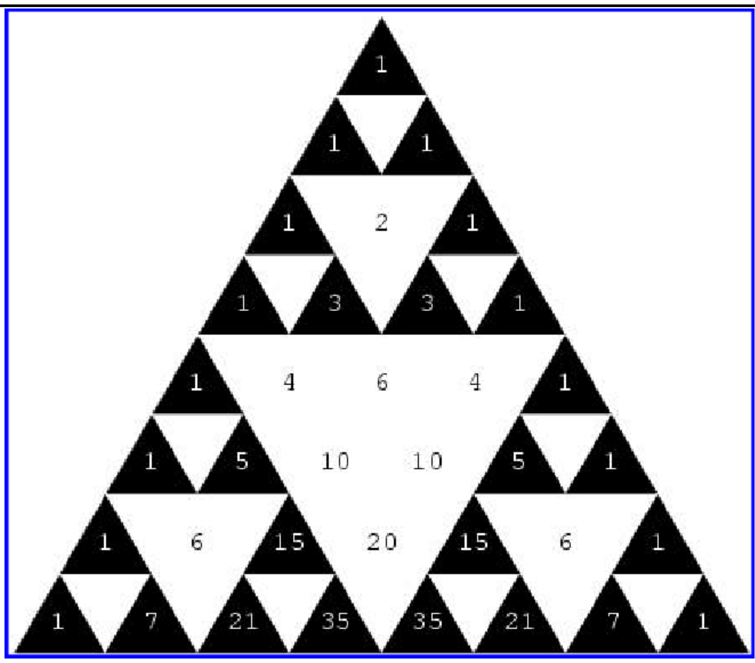
$_4C_1$ → $t_{4,1}$, $_2C_1$ → $t_{2,1}$

Terms can be formed using the combinations formula

Generate the next row by adding adjacent terms and writing the answer in the gap underneath.

Note: There are zeros to the other sides of the 1s.


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1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1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
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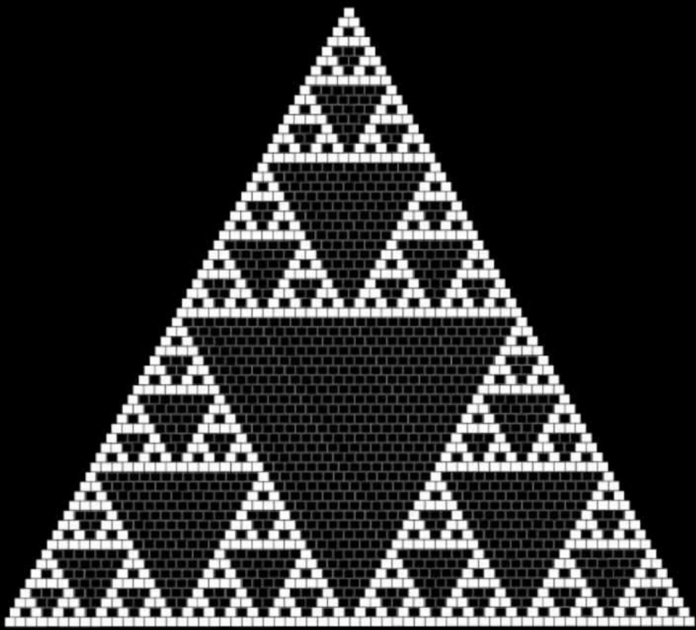


Example 1

Diagonal Patterns in Pascal's Triangle

Calculate the sum of the first four terms of diagonal 2. Find this sum in Pascal's triangle and relate it to $_{n}C_{r}$.

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

330

Diagonal 2 $\Rightarrow$ 1, 3, 6, 10, ...

Sum of first 4 terms $= 1+3+6+10$
$= 20$

This is also the same as

$_{6}C_{3}$ $\quad [_{2}C_{2} + {_{3}C_{2}} + {_{4}C_{2}} + {_{5}C_{2}}]$

Gives the "Hockey Stick" pattern

Your Turn

Calculate the sum of the first five terms of diagonal 6. Find this value in Pascal's triangle and relate it to $_{n}C_{r}$.

Diagonal 6 $\Rightarrow$ 1, 7, 28, 84, ...

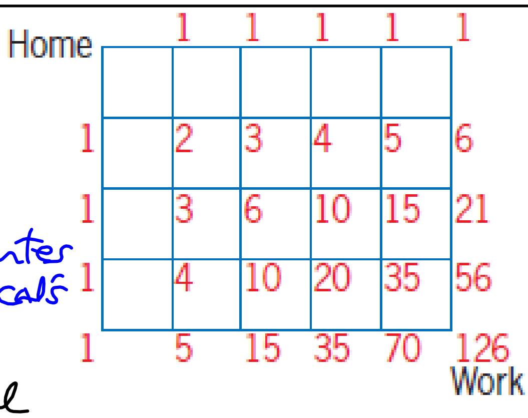
Sum of first 5 terms $= 1+7+28+84+210$
$= 330$

This is also the same as $_{11}C_{7}$

Example 2

Routes on a Grid

To get from home to work, Hannah travels four blocks south and five blocks east. How many different routes can she take, travelling only south and east?



The number at each corner represents the number of routes to get there. It follows Pascal's triangle!

OR Travelling 9 blocks in total

$\Rightarrow {}_9C_4 \times {}_5C_5$ (south, East) $= 126 \times 1$

$= 126$ routes

Your Turn

Bill rides his bike to school and travels four blocks west and six blocks north. Use two methods to determine the number of different routes Bill could take to school if he travels only north and west.

Travelling 10 blocks in total

$\Rightarrow {}_{10}C_4 \times {}_6C_6$

$= 210 \times 1$

$= 210$

Example 3

Pascal's Triangle and the Binomial Theorem

a) Expand and simplify $(x + y)^2$ and $(x + y)^3$.

b) Relate the coefficients in the binomial expansion to Pascal's triangle and to combinations.

c) State the degree of each term and describe the pattern in the exponents.

a) $(x+y)^2$

$= (x+y)(x+y)$

$= x^2 + xy + xy + y^2$

$= x^2 + 2xy + y^2$

$(x+y)^3$

$= (x+y)(x+y)(x+y)$

$= (x^2+2xy+y^2)(x+y)$

$= (x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$

$= x^3 + 3x^2y + 3xy^2 + y^3$

b) Coefficients relate to the numbers in a given row for Pascal's triangle.

$1x^2 + 2xy + 1y^2$ are the terms in row 2

$1x^3 + 3x^2y + 3xy^2 + 1y^3$ are the terms in row 3

Combinations relate to ${}_nC_r$ where n is the row number and r is the position of the term.

c) Each successive term has the exponent of x decrease by one, whilst the exponent of y increases by one.

Your Turn

Use the binomial theorem to expand each binomial, relating it to both Pascal's triangle and combinations. State the degree of each term.

a) $(a + b)^4$ b) $(p + q)^5$

Watch the binomial theorem video for help with these.

a) $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

b) $p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

Key Concepts

- The terms in row n of Pascal's triangle correspond to the combinations $t_{n,r} = {}_nC_r$.

https://www.youtube.com/watch?v=XMriWTvPXHI

- A given term in Pascal's triangle equals the sum of the two terms directly above it in the previous row. They can be generated using the relationship $t_{n,r} + t_{n,r+1} = t_{n+1,r+1}$. This is known as Pascal's method.
- Using combinations, ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$.
- The coefficients in the binomial expansion of $(x + y)^n$ are found in row n of Pascal's triangle.
- According to the binomial theorem, $(x + y)^n = {}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_{n-1}x^1y^{n-1} + {}_nC_nx^0y^n$.

The general term is ${}_nC_rx^{n-r}y^r$.

- Pascal's method can be applied to counting paths in arrays.

R2. Describe how Pascal's triangle and combinations are related. Use one row as a reference.

The terms of row n in Pascal's triangle correspond to the combinations $t_{n,r} = {}_nC_r$. Each row in Pascal's triangle represents the combinations of choosing 0 items, 1 item, 2 items, and so on, out of n items.