Problem Solving with Combinations

Lesson objectives

- I can distinguish situations that use permutations from those that use combinations
- I can solve counting problems using the rule of sum and the fundamental counting principle

1.1

Lesson objectives | Teachers' notes | Lesson notes

MHR Page 120 #s 1 - 5, 6 - 8 & 10

Warm Up

You have learned techniques for counting items in which order does and does not matter. How many possible sums are there if you have

- one nickel and one toonie
- three dimes and two quarters

How can you be sure you did not miss any amounts? Explain.

1 Nickel or 1 Toonie or 1 Nick, 1 Toon

\$0.05, \$2.00, \$2.05

3 dimes, 2 quarters

Selecting ONE coin

$\Rightarrow$ 1 dime or 1 quarter

Selecting TWO coins

$\Rightarrow$ 2 dimes, 2 quarters, 1 dime and 1 quarter

Selecting THREE coins

$\Rightarrow$ 3 dimes, 2 dimes and 1 quarter, 1 dime and 2 quarters

Selecting FOUR coins

3 dimes and 1 quarter, 2 dimes and 2 quarters

Selecting FIVE coins

3 dimes and 2 quarters

Total of 11 different sums of money.

Definition

Subset

- A set whose elements are also elements of another set.

Null Set

- A set with no elements.

Total Number of Subsets of a Set

When building subsets of a given set, you can choose from 0 to n elements to be in the subset. The total number of subsets of a set of n elements is ${}_nC_0 + {}_nC_1 + \ldots + {}_nC_n$.

Another way of looking at this is that each given element of a set could be either included or excluded, which can be counted as two different ways. Since there are n elements, this could be done in $2 \times 2 \times 2 \ldots \times 2 = 2^n$ ways. The total number of subsets of a set of n elements is 2^n.

Null sets have zero elements. If the null set is excluded from the tally, the total number of subsets would be $2^n - 1$.

Example 1

Determine the Total Number of Subsets

Apples, grapes, peaches, plums, and strawberries are available for dessert.
How many different combinations of fruit can be made for dessert?

Have to choose at least one fruit

$\Rightarrow {}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5$

$= 5 + 10 + 10 + 5 + 1$

$= 31$ combinations

OR use indirect method
There are 5 different fruits. We can either choose a fruit, or not choose it.

$\Rightarrow 2^5 = 32$ combinations.

However, we must choose at least one fruit, so we have to subtract the null-set.

$\Rightarrow 2^5 - 1$

$= 32 - 1 = 31$ combinations

Your Turn

A bag contains eight different-coloured marbles. Use two methods to determine the number of ways to reach into the bag and pull out one or more marbles.

$_8C_1 + {}_8C_2 + {}_8C_3 + {}_8C_4 + {}_8C_5 + {}_8C_6 + {}_8C_7 + {}_8C_8$

$= 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1$

$= 255$ ways

OR use indirect method

each marble can be chosen, or not chosen

$\Rightarrow 2^8 = 256$

We must choose at least one marble so we need to subtract the null-set.

$\Rightarrow 2^8 - 1$

$= 256 - 1 = 255$ ways

Example 2

Count Cases

The card game euchre uses only the 9s, 10s, jacks, queens, kings, and aces. Five-card hands are dealt to the players. How many euchre hands contain

a) at least three queens?

b) at least two black cards?

24 cards are used.
4 9's, 4 10's, 4 J's, 4 Q's, 4 K's & 4 A's.

Case 1 $\Rightarrow$ 3 Queens

$\Rightarrow {}_4C_3 \times {}_{20}C_2 = 4 \times 190 = 760$

3 Queens from 4

2 cards from the other 20

Case 2 $\Rightarrow$ 4 Queens

$\Rightarrow {}_4C_4 \times {}_{20}C_1 = 1 \times 20 = 20$

4 Queens from 4

1 card from the other 20

Total = 780 hands

Example 2

Count Cases

The card game euchre uses only the 9s, 10s, jacks, queens, kings, and aces. Five-card hands are dealt to the players. How many euchre hands contain

a) at least three queens?

b) at least two black cards?

Use the indirect method

All combinations $= {}_{24}C_5$
$= 42{,}504$

Case 1 $\Rightarrow$ No black cards

$\Rightarrow {}_{12}C_0 \times {}_{12}C_5 = 1 \times 792 = 792$

0 black from 12

5 reds from the 12 reds

Case 2 $\Rightarrow$ 1 black card

$\Rightarrow {}_{12}C_1 \times {}_{12}C_4 = 12 \times 495 = 5940$

1 black from 12

4 red from 12 reds

Total hands of at least 2 black cards

= All hands − zero black − one black

= 42,504 − 792 − 5940 = 35,772 hands

Your Turn

In the game of hearts, the entire deck of cards is dealt. If you have a hand with 13 cards, in how many ways could the hand contain

a) at least two hearts?

b) at least ten hearts?

c) five clubs and five spades?

d) three diamonds?

e) five clubs or five spades?

Total combinations to be dealt $= {}_{52}C_{13}$

a) Zero hearts

$= {}_{13}C_0 \times {}_{39}C_{13}$

one heart

$= {}_{13}C_1 \times {}_{39}C_{12}$

$\Rightarrow$ At least 2 hearts

$= {}_{52}C_{13} - ({}_{13}C_0 \times {}_{39}C_{13}) - ({}_{13}C_1 \times {}_{39}C_{12})$

$= 5.760507675 \times 10^{11}$

c) 5 clubs AND 5 spades

$= {}_{13}C_5 \times {}_{13}C_5 \times {}_{26}C_3$

$= 1287 \times 1287 \times 2600$

$= 4{,}306{,}559{,}400$

b) 10 hearts

$= {}_{13}C_{10} \times {}_{39}C_3$

11 hearts

$= {}_{13}C_{11} \times {}_{39}C_2$

12 hearts

$= {}_{13}C_{12} \times {}_{39}C_1$

13 hearts

$= {}_{13}C_{13} \times {}_{39}C_0$

$\Rightarrow$ At least 10 hearts

= sum of 10H, 11H, 12H, 13H

= 2,672,060

Your Turn

In the game of hearts, the entire deck of cards is dealt. If you have a hand with 13 cards, in how many ways could the hand contain

a) at least two hearts?

b) at least ten hearts?

c) five clubs and five spades?

d) three diamonds?

e) five clubs or five spades?

d) 3 diamonds

$= {}_{13}C_3 \times {}_{39}C_{10}$

$= 286 \times 635,745,396$

$= 1.81823183 \times 10^{11}$

e) 5 clubs OR 5 spades

$= {}_{13}C_5 \times {}_{39}C_8 + {}_{13}C_5 \times {}_{39}C_8$

$= 1287 \times 61,523,748 + 1287 \times 61,523,748$

$= 6.269640845 \times 10^{21}$

Example 3

Choose, Then Arrange

Christine has 10 pictures of family and 8 pictures of friends to put on her wall. She installs shelves to display the pictures. The shelves can fit only four of the family pictures and three of the friends pictures. In how many ways can Christine arrange the pictures on the shelves?

First choose the pictures

$\Rightarrow {}_{10}C_4 \times {}_8C_3$

$= 210 \times 56$

$= 11760$

These 7 pictures can be arranged in 7! ways

$\Rightarrow n(A) = 11760 \times 7!$

$= 59,270,400$ ways

Your Turn

a) How many five-letter codes can be formed from two different vowels and three different consonants? Consider Y a vowel.

b) How many of these codes contain the letter C?

a) 6 "vowels", 20 consonants

$\Rightarrow {}_6C_2 \times {}_{20}C_3 \times 5!$ arrange the letters

$= 15 \times 1140 \times 120$

$= 2{,}052{,}000$ codes

b) If you have to have a "C" then

$\Rightarrow {}_6C_2 \times {}_1C_1 \times {}_{19}C_2 \times 5! = 15 \times 1 \times 171 \times 120$

$= 307{,}800$ codes

vowels "C" other consonants

Key Concepts

- The total number of subsets of a set of n elements is ${}_nC_0 + {}_nC_1 + \ldots + {}_nC_n = 2^n$.
- In some cases the null set is not considered. In such cases, ${}_nC_1 + {}_nC_2 + \ldots + {}_nC_n = 2^n - 1$.
- Consider using the indirect method, especially if it involves fewer cases, such as when you need to choose at least one or two items.
- If the order is important, consider selecting the items first and then arranging them in order.

R1. When determining the total number of subsets of a set, you add the number of possibilities in each case. Explain why you add instead of multiply.

When determining the total number of subsets of a set, you add the number of possibilities in each case because the events are mutually exclusive.

R2. When using cases to determine the number of ways of selecting objects from different sets, do you multiply or add? Explain your reasoning.

When using cases to determine the number of ways of selecting objects from different sets, you add because the events are mutually exclusive.

R3. You can solve counting problems using powers, permutations, combinations, or both. Make a summary and a flowchart to help decide which method(s) to use. Include simple examples to support your summary.