Permutations with Non-Ordered Elements

Lesson objectives

- I can recognise the advantages of different counting techniques
- I can make connections between situations that involve permutations and combinations

1.1

Lesson objectives | Teachers' notes | Lesson notes

MHR Page 108 #s 1 - 9 & 13

Warm Up

In the previous chapter, the objects in each set were always different. This is not always the case. Sometimes, there are objects that are identical.

- Do you think this will increase (or decrease) the number of different arrangements? Why?

Decrease, because there will be some repeated arrangements.

- There may be some objects that cannot be rearranged. Do you think this will increase (or decrease) the number of arrangements? Why?

Decrease. If some objects are in fixed positions there are fewer objects that can be arranged.

Complete the Investigate on Page 104

1. Select the coloured blocks as indicated. Set the others aside. For each set of blocks, answer the following questions:
   - What are the possible permutations?
   - How many arrangements of blocks in a row are there?

   Different looking arrangements

   6 coloured blocks or linking cubes (3 of one colour, 2 of a second colour, 1 of a third colour)

   a) two of one colour, one of another colour

   Permutations = 3! = 6

   Blocks in a row = 3

   b) two of one colour, two of different colours (e.g., blue blue yellow red)

   Permutations = 4! = 24

   Blocks in a row = 12

   c) two of one colour, two of another colour

   Permutations = 4! = 24

   Blocks in a row = 6

   d) two of one colour, three of another colour

   Permutations = 5! = 120

   Blocks in a row = 10

Complete the Investigate on Page 104

1. Select the coloured blocks as indicated. Set the others aside. For each set of blocks, answer the following questions:
   - What are the possible permutations?
   - How many arrangements of blocks in a row are there?

   6 coloured blocks or linking cubes (3 of one colour, 2 of a second colour, 1 of a third colour)

   a) two of one colour, one of another colour

   RRB RBR BRR

   3 distinct arrangements

   b) two of one colour, two of different colours (e.g., blue blue yellow red)

   12 distinct arrangements

   RRBG | RGRB | BRRG | GRRB
   RRGB | RGBR | BRGR | GRBR
   RBRG | RBGR | BGRR | GBRR

   c) two of one colour, two of another colour

   RRBB BBRR RBRB
   RBBR BRRB BRBR

   6 distinct arrangements

   d) two of one colour, three of another colour

   BBRRR BRBRR BRRBR
   BRRRB RBRRB RRBRB
   RRRBB RRBBR RBRBR
   RBBRR

   10 distinct arrangements

Example 1

Permutations With Like Elements

Compare the number of arrangements of the sets of letters A_1A_2BC and AABC.

A_1A_2BC would have 24 arrangements.
A_1 is distinguishable from A_2 $(4!)$
$\Rightarrow$ order is important.

AABC would have 12 arrangements.
A and A are identical objects and can be arranged in 2! ways. $\left(\frac{4!}{2!}\right)$

$\Rightarrow$ order is not important.

Your Turn

Compare the number of arrangements of the sets of letters.

a) $AB_1B_2B_3$ and ABBB

b) $A_1A_2B_1B_2$ and AABB

a) $AB_1B_2B_3$ are different objects
$\Rightarrow 4! = 24$ arrangements.
ABBB has 3 identical objects.
$\Rightarrow \frac{4!}{3!} = 4$ arrangements

b) $A_1A_2B_1B_2$ are different objects
$\Rightarrow 4! = 24$ arrangements.
AABB has 2 pairs of identical objects.
$\Rightarrow \frac{4!}{2!2!} = 6$ arrangements

Permutations With Like Objects

In Example 1, you can multiply the number of arrangements of AABC by 2! to determine the number of arrangements of A_1A_2BC. You can also divide the number of arrangements of A_1A_2BC by 2! to determine the number of arrangements of AABC.

The number of permutations of n elements, when p of one type are identical, q of another type are identical, r of another type are identical, and so on, is $n(A) = \frac{n!}{p!q!r!\ldots}$.

Try this one: B, B, B, G, G, R, R, R

8 elements
3 B's
2 G's
3 R's

$$\Rightarrow n(A) = \frac{\overset{\#\text{ elements}}{8!}}{\underset{\#B\text{'s}}{3!} \times \underset{\#G\text{'s}}{2!} \times \underset{\#R\text{'s}}{3!}}$$

$$= \frac{40{,}320}{72} = 560 \text{ arrangements}$$

Example 2

Permutations With Several Identical Elements

A hockey team ended its season with 12 wins, 8 losses, and 4 ties. In how many orders could these outcomes have happened?

Total of 12 + 8 + 4 = 24 games.

$$n(A) = \frac{\overset{\text{games}}{24!}}{(\underset{\text{wins}}{12!} \times \underset{\text{losses}}{8!} \times \underset{\text{ties}}{4!})} = 1{,}338{,}557{,}220$$

Your Turn

Compare the number of orders of this hockey team's wins, losses, and ties with those of a team that had eight wins, eight losses, and eight ties. Would you expect the number of orders to be higher or lower in the second scenario? Why?

$$n(A) = \frac{24!}{8! \times 8! \times 8!} = 9{,}465{,}511{,}770$$

I would expect it to be higher in the second scenario because the denominator is smaller, which leads to a bigger answer.

$$12(11)(10)(9)(8!)(8!)(4!)$$
vs
$$8(7)(6)(5)(4!)(8!)(8!)$$

Example 3

Distinct Objects in a Fixed Order

How many ways are there to arrange the letters in the word NUMBER if the consonants must remain in the original order?

As the consonants cannot change order we can treat them as identical objects.

$$n(A) = \frac{6!}{4!} = 30 \text{ ways.}$$

\# of letters

\# of consonants

Your Turn

How many permutations are there of the letters in the word EXPLAIN if the vowels must be in alphabetical order?

Again we can treat the vowels as identical objects because they must remain in order.

$$n(A) = \frac{7!}{3!} = 840$$

\# of letters

\# of vowels

Key Concepts

- The number of permutations of n objects, when p of one type are identical, q of another type are identical, r of another type are identical, and so on, is $n(A) = \frac{n!}{p!q!r!\ldots}$.
- If a number of distinct objects need to remain in a specific order in a permutation, divide by the factorial of that number.

R1. Explain why you need to divide by 4! when calculating the number of arrangements of the digits 1, 2, 2, 2, 2, 3, 4.

You must divide by 4! because you have four identical 2s.

R2. Is the number of permutations of three girls and four boys the same as the number of permutations of three red balls and four green balls? Explain.

No! The permutations for 3 girls and 4 boys is 7!. They are individuals, not identical people. The permutations for the balls would be 7!/(3!4!)

R3. Why is it easier to use the formula $n(A) = \frac{n!}{p!q!r!\ldots}$ than to use a tree diagram or chart?

It is quicker to use the formula. Identify the number of objects and the number of identical objects. Tree diagrams and charts can get complicated and impractical.