Solutions

1. Evaluate.

a) 8!

b) $\frac{9!}{3!}$

c) $3! \times 4!$

d) ${}_{10}P_6$

e) ${}_{12}P_3$

f) $\frac{{}_7P_3}{3!}$

g) $\frac{{}_{11}P_4}{4!}$

h) $\frac{14!}{2!5!6!}$

a) $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$= 40,320$

b) $\frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!}}$
$= 60,480$

c) $3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1$
$= 144$

d) ${}_{10}P_6$
$= \frac{10!}{(10-6)!}$
$= 151,200$

e) ${}_{12}P_3$
$= \frac{12!}{(12-3)!}$
$= 1320$

f) $\frac{{}_7P_3}{3!} = \frac{7!}{(7-3)!3!}$
$= \frac{7!}{4!3!} = 35$

g) $\frac{{}_{11}P_4}{4!} = \frac{11!}{(11-4)!4!}$
$= \frac{11!}{7!4!} = 330$

h) $\frac{14!}{2!5!6!} = 504,504$

2. a) Define $n!$ in words and with a formula.

n! (factorial) is a product of sequential natural numbers with the form n(n-1)(n-2) x ... x 3 x 2 x 1.

b) Define $_nP_r$ in words and with a formula.

The number of permutations of *r* items from a collection of *n* items is written as $_nP_r$ or P(n,r)

$$_nP_r = \frac{n!}{(n-r)!}, \quad n \geq r$$

3. Express each permutation in factorial form.

a) $_7P_3$

b) $_{100}P_{92}$

c) $_nP_6$

d) $_{15}P_r$

a) $\frac{7!}{(7-3)!}$

$= \frac{7!}{4!}$

b) $\frac{100!}{(100-92)!}$

$= \frac{100!}{8!}$

c) $\frac{n!}{(n-6)!}$

d) $\frac{15!}{(15-r)!}$

4. How many ways are there to arrange
   a) 8 objects?
   b) 5 of 8 objects?
   c) 3 of 13 objects?

a) $8 \times 7 \times 6 \times 5 \times \ldots$
$= 8!$
$= 40,320$

b) ${}_8P_5$
$= \frac{8!}{(8-5)!}$
$= \frac{8!}{3!}$
$= 6720$

c) ${}_{13}P_3$
$= \frac{13!}{(13-3)!}$
$= \frac{13!}{10!}$
$= 1716$

5. a) In how many ways could five girls and six boys line up in one row?

Total of 11
$\Rightarrow n(A) = 11!$
$= 39,916,800$

b) In how many ways could they line up with the girls in the front row and the boys in the back row?

Boys $= 6$
$= 6!$
$= 720$
Girls $= 5$
$= 5!$
$= 120$
We want boys AND girls $\Rightarrow$ MULTIPLY
$= 720(120)$
$= 86,400$

6. a) How many permutations are there of all the letters in TRIANGLE?

8 letters
All are different

$\Rightarrow 8!$

$= 40,320$

b) How many arrangements are there of any three of the letters in TRIANGLE?

$\Rightarrow {}_8P_3$

$= \frac{8!}{(8-3)!}$

$= \frac{8!}{5!}$

$= 336$

7. This array of numbers is called Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

a) Investigate the terms and describe how to determine the next row. Continue Pascal's triangle for two more rows.

b) Describe any patterns you see in the triangle.

a) Two adjacent numbers add to make the number below/between them.

b) Counting numbers
Triangular numbers
Tetrahedral numbers
Sum of rows = powers of 2
Reading rows = powers of 11

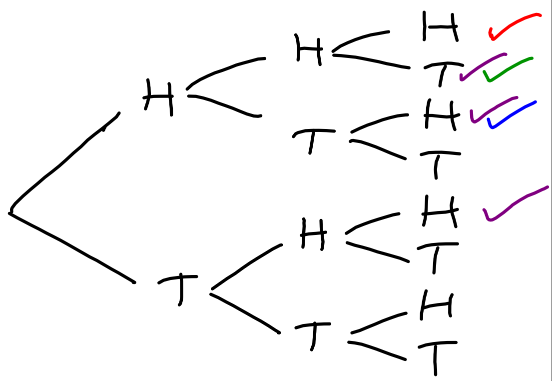
8. A coin is flipped three times. Calculate the probability of each event.

a) heads, heads, heads

b) heads, heads, tails

c) heads, tails, heads

d) two heads and one tail in any order



a) $P(HHH) = \frac{1}{8}$

b) $P(HHT) = \frac{1}{8}$

c) $P(HTH) = \frac{1}{8}$

d) $P(2\text{ Hs}, 1\text{ T any order}) = \frac{3}{8}$

9. Two cards are dealt from a standard deck. Determine each probability.

a) The first card is a king and the second card is an ace.

b) The first card is red and the second card is black.

c) The first card is a heart or the second card is a king.

52 cards in a deck

a) $P(KA) = \frac{4}{52} \times \frac{4}{51}$
$= \frac{16}{2652}$
$= \frac{4}{663} \quad (\approx 0.006)$

b) $P(RB) = \frac{26}{52} \times \frac{26}{51}$
$= \frac{676}{2652}$
$= \frac{13}{51} \quad (\approx 0.2549)$

c) $P(H \text{ or } K) = P(H) + P(K) - P(HK)$
$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$
$= \frac{16}{52}$
$= \frac{4}{13} \quad (\approx 0.3077)$

10. There are 18 students in a class. Their names are drawn, at random, to determine the order in which they will present their projects. Determine each probability.

a) Jacob is first.

b) Caryn is last.

c) Jacob is first or Caryn is last.

d) The names are in alphabetical order.

a) $P(\text{Jacob } 1^{ST}) = \frac{1}{18}$

b) $P(\text{Caryn Last}) = \frac{1}{18}$

c) $P(\text{Jacob } 1^{ST} \text{ or Caryn last}) = \frac{1}{18} + \frac{1}{18}$

$= \frac{2}{18}$

$= \frac{1}{9}$

d) $P(\text{Alphabetical}) = \frac{1}{18} \times \frac{1}{17} \times \frac{1}{16} \times \ldots \times \frac{1}{2} \times \frac{1}{1}$

$= \frac{1}{18!}$

$= \frac{1}{6.402 \times 10^{15}}$

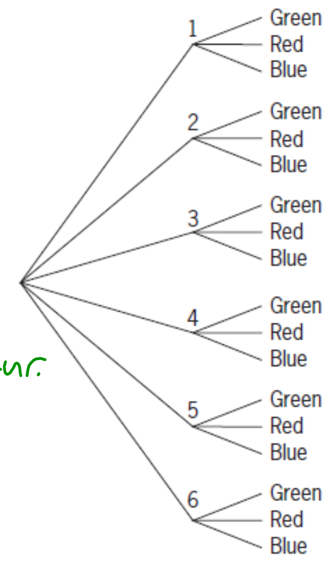
11. The tree diagram illustrates all the possible outcomes when a standard die and a coloured die are rolled.

a) Describe the faces of the coloured die.

b) How many different outcomes are there?

c) Determine the probability $P(5, \text{Red})$.

d) What is the probability $P(\text{Green or Blue})$?



a) Only 3 colour choices.
Probably a 6-sided die
with two faces of each colour.

b) 6 numbers × 3 colours
= 18 outcomes

c) $P(5, \text{Red}) = \frac{1}{6} \times \frac{1}{3}$

$= \frac{1}{18}$

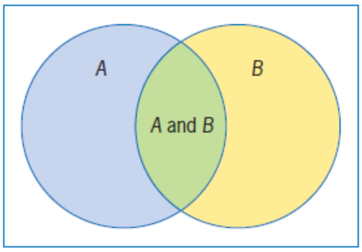
d) $P(\text{Green or Blue}) = P(\text{Green}) + P(\text{Blue})$

$= \frac{1}{3} + \frac{1}{3}$

$= \frac{2}{3}$

Doesn't matter what the number is

12. Use the Venn diagram to help you explain the principle of inclusion and exclusion.



The events A and B are **NOT** mutually exclusive as there is an overlap that represents common elements.

If A and B are non-mutually exclusive then the total number of favourable outcomes is :

$$n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$$

13. In a game of cards, a hand contains five red cards and four face cards. Two cards are red face cards. How many cards are in the hand?

$$n(\text{Cards}) = n(\text{Red}) + n(\text{Face}) - n(\text{Red Face})$$
$$= 5 + 4 - 2$$
$$= 7 \text{ cards}$$

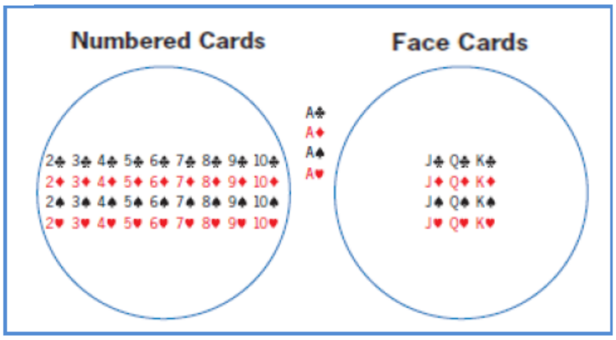
14. Draw a Venn diagram to illustrate each of the following.

a) Face cards and numbered cards from a standard deck.

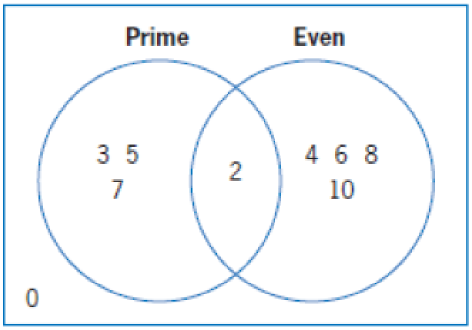
b) Prime and even whole numbers. Remember, 0 is neither prime nor even.

c) Vowels and consonants. What can be said about "Y"?

d) The set of integers and the set of natural numbers.



All cards are in the sample space. The subsets are face cards and numbered cards. They are mutually exclusive so there is no overlap. The aces are in neither subset.



The sample space is infinite, but for ease I have chosen the integers from 0 to 10. The subsets are prime numbers and even numbers. They are **NOT** mutually exclusive so there is an overlap. Zero is neither prime nor even.

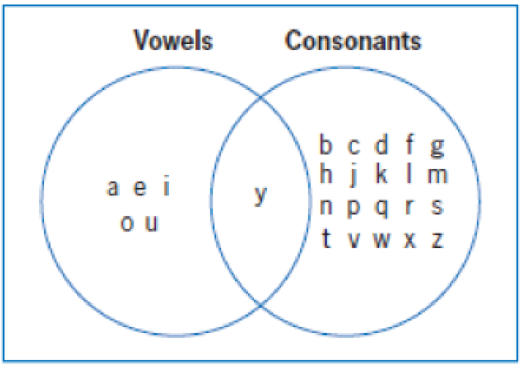
14. Draw a Venn diagram to illustrate each of the following.

a) Face cards and numbered cards from a standard deck.

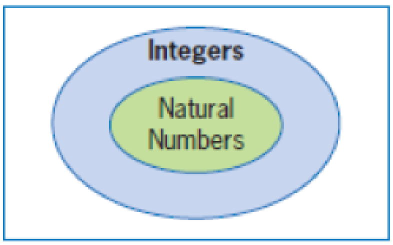
b) Prime and even whole numbers. Remember, 0 is neither prime nor even.

c) Vowels and consonants. What can be said about "Y"?

d) The set of integers and the set of natural numbers.



The sample space is the letters of the alphabet. The subsets are vowels and consonants. They are **NOT** mutually exclusive as "y" is a consonant but acts as a vowel so there is an overlap. All letters are accounted for so there are no letters outside of the subsets.



The sample space is the real numbers. The subsets are the integers and the natural numbers (positive integers). They are **NOT** mutually exclusive as all natural numbers are integers. This means that the natural numbers are a subset of the integers.

15. Simplify.

a) $(x^2)^3$

b) $(2a)^2$

c) $(5m^3)^2$

d) $(3k^3)^4$

a) $(x^2)^3$
$= x^{2\times3}$
$= x^6$

b) $(2a)^2$
$= (2)^2(a^{1\times2})$
$= 4a^2$

c) $(5m^3)^2$
$= (5)^2(m^{3\times2})$
$= 25m^6$

d) $(3k^3)^4$
$= (3)^4(k^{3\times4})$
$= 81k^{12}$

16. Expand and simplify.

a) $(x + y)^2$

$= (x+y)(x+y)$
$= x^2 + xy + xy + y^2$
$= x^2 + 2xy + y^2$

b) $(a + b)^3$

$= (a+b)(a+b)(a+b)$
$= (a^2 + ab + ab + b^2)(a+b)$
$= (a^2 + 2ab + b^2)(a+b)$
$= a^3 + 2a^2b + ab^2$
$+ a^2b + 2ab^2 + b^3$
$= a^3 + 3a^2b + 3ab^2 + b^3$

c) $(2p + q)^2$

$= (2p+q)(2p+q)$
$= 4p^2 + 2pq + 2pq + q^2$
$= 4p^2 + 4pq + q^2$

17. Simplify by reducing.

a) $\frac{(n-1)(n-2)(n-3)(n-4)}{(n-3)(n-4)}$

$= \frac{(n-1)(n-2)\cancel{(n-3)}\cancel{(n-4)}}{\cancel{(n-3)}\cancel{(n-4)}}$

$= (n-1)(n-2)$

b) $\frac{n(n-1)(n-2)\cdots 3\times 2\times 1}{(n-2)(n-3)(n-4)\cdots 3\times 2\times 1}$

$= \frac{n(n-1)\cancel{(n-2)}\cdots\cdots\cancel{(3)}\cancel{(2)}\cancel{(1)}}{\cancel{(n-2)}\cancel{(n-3)}\cancel{(n-4)}\cdots\cancel{(3)}\cancel{(2)}\cancel{(1)}}$

$= n(n-1)$

c) $\frac{n!}{(n-1)!}$

$= \frac{(n)\cancel{(n-1)!}}{\cancel{(n-1)!}}$

$= n$