Permutations and Factorials

Lesson objectives

- I see how using permutations have advantages over other counting techniques
- I can solve simple problems using techniques for counting permutations
- I can write permutation solutions using proper mathematical notation

1.1

Lesson objectives | Teachers' notes | Lesson notes

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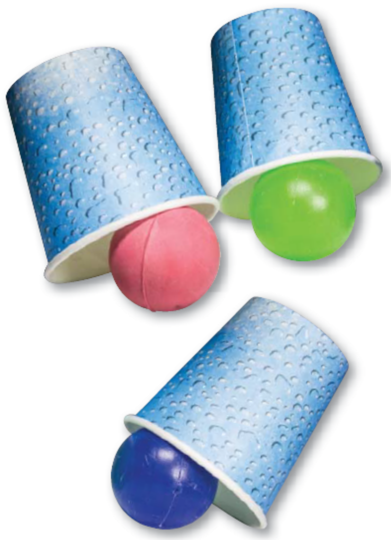
Warm Up

A game involves placing three different coloured balls under three different cups, then quickly mixing them up. The player must correctly identify the final order of the three colours. In how many different orders could the colours finish? Describe at least two ways to figure this out.



3 outcomes for 1ST cup

2 outcomes for 2ND cup

1 outcome for 3RD cup

$\Rightarrow 3 \times 2 \times 1 = 6$ ways

Complete the "Investigate" on Page 76

Definitions

Arrangement

- An ordered list of items

Factorial

- A product of sequential natural numbers with the form

$n! = n(n - 1)(n - 2)... \times 2 \times 1$

- n! is read "n factorial"

Permutation

- An arrangement of n distinct items in a definite order
- The total number of these permutations is written $_nP_n$ or

P(n,n)

Example 1

Evaluating Factorial Expressions

Evaluate each **factorial**.

a) 3! b) 5! c) 10! d) $\frac{6!}{4!}$

a) $3! = 3 \times 2 \times 1 = 6$

b) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

c) $10! = 10 \times 9 \times 8 \times 7 \times ...$
$= 3628800$

d) $\frac{6!}{4!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$
$= 6 \times 5 = 30$

Your Turn

Evaluate each factorial.

a) 4! b) 6! c) $\frac{11!}{7!}$ d) $\frac{6! \times 4!}{5!}$

a) $4! = 4 \times 3 \times 2 \times 1 = 24$

b) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$= 720$

c) $\frac{11!}{7!} = 11 \times 10 \times 9 \times 8$
$= 7920$

d) $\frac{6! \times 4!}{5!} = 6 \times 4! = 144$

When an arrangement of items needs to appear in order, it is called a permutation. There are n ways of selecting the first item, (n - 1) ways of selecting the second item, (n - 2) ways of selecting the third and so on, until there is one way left of selecting the last item. Using the fundamental counting principle, we multiply these numbers together:

$$n(n - 1)(n - 2)(n - 3) \ldots \times 3 \times 2 \times 1, \text{ which is n factorial.}$$

The number of permutations of n items is ${}_nP_n = n!$

Example 2

Counting Permutations

A photographer lines up six people. How many different arrangements could she make?

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Your Turn

A half-hour TV show has eight 30-second advertisement time slots. In how many ways could the eight advertisements be assigned a time?

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$

Example 3

Permutations of Some Items in a Set

There are 12 people on a swim team. Four will be chosen to take part in a relay, racing in a given order. In how many ways could the four swimmers be selected?

$= \dfrac{\text{Total arrangements}}{\text{Arrangements of NOT chosen}}$

$= \dfrac{12!}{8!} = 11880$

OR

Choose from 12
Choose from 11
Choose from 10
Choose from 9
$= 12 \times 11 \times 10 \times 9$
$= 11880$

Your Turn

Forty athletes are entered in a triathlon. Medals are presented to the top three finishers. In how many ways could the gold, silver, and bronze medals be awarded?

Gold = 40
Silver = 39
Bronze = 38
$= 40 \times 39 \times 38$
$= 59,280$

OR

$\dfrac{\text{Total arrangements}}{\text{Arrangements of NOT medals}}$

$= \dfrac{40!}{37!} = 59,280$

Example 4

Permutations With Restrictions

A librarian wants to display 10 books by Canadian authors on a bookshelf. There are three books by Joseph Boyden, and the rest are by different authors. In how many ways could he arrange the books if the Joseph Boyden books must remain side-by-side?

Treat 3 Boyden books as one selection

$\Rightarrow$ Arranging "8" books and the 3 Boyden books

$= 8! \times 3! = 241{,}920$

Your Turn

Six team photos are hanging on the wall outside a high school gym. Two of the photos are of the junior and senior football teams. In how many ways could they be arranged in a straight line if the two football photos must be beside each other?

Treat 2 football photos as one Selection

$\Rightarrow$ Arranging "5" photos and the 2 football photos

$= 5! \times 2! = 240$

Key Concepts

- The number of permutations of n items is n factorial,

  $n! = n(n-1)(n-2)(n-3) \ldots \times 3 \times 2 \times 1$
- You can use factorials as a counting technique when repetition is not permitted.
- The number of r-permutations of n items can be calculated by

  $${}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$$
  $$= \frac{n!}{(n-r)!}$$

Homework

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